Lecture 6 The Bayes' Theorem

BIO210 Biostatistics

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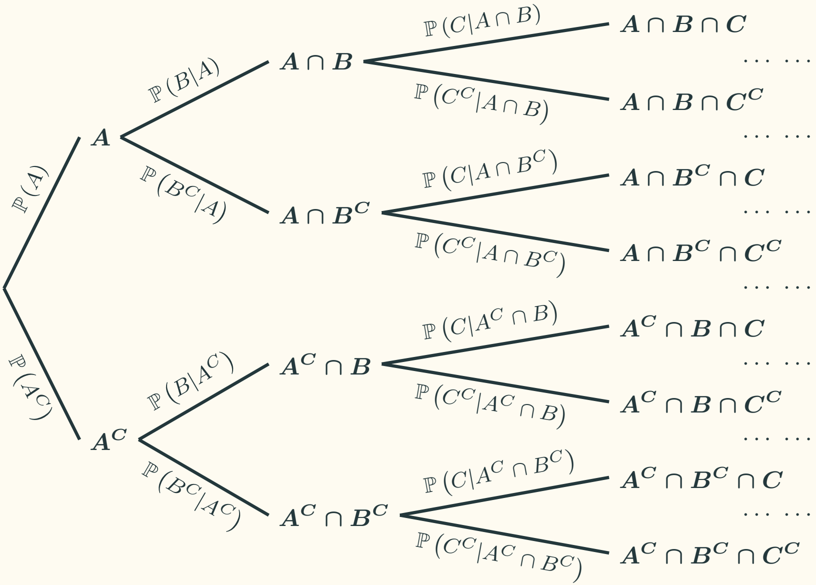
Basic components

Three basic components in conditional probability:

1. $\mathbb{P}(A \cap B)$
2. $\mathbb{P}(B)$
3. $\mathbb{P}(A|B)$

Generalisation of $\mathbb{P}(A \cap B \cap C \cap \cdots)$



The multiplication rule:

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) =$$

$\mathbb{P}(A_1)\cdot$

$\mathbb{P}(A_2|A_1)\cdot$

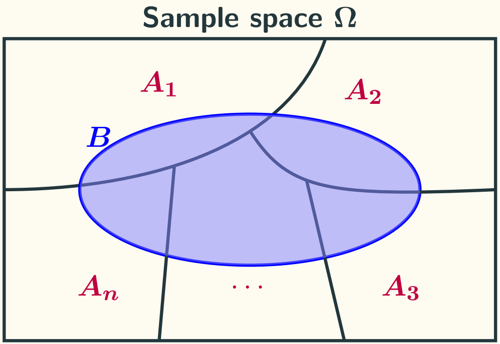
$\mathbb{P}(A_3|A_1 \cap A_2)\cdot$

$\mathbb{P}(A_4|A_1 \cap A_2 \cap A_3)\cdot$

$\cdots$

$$\mathbb{P}\left(A_n|\bigcap_{i=1}^{n-1} A_i\right)$$

Generalisation of $\mathbb{P}(B)$

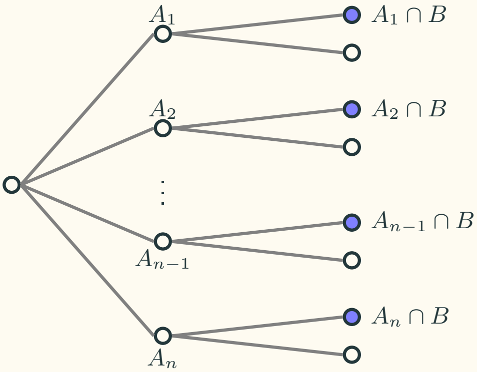


$$
\begin{aligned}
\mathbb{P}(B) =& \mathbb{P}[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)] \\
=& \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \cdots + \mathbb{P}(A_n \cap B) \\
=& \mathbb{P}(A_1)\,\mathbb{P}(B|A_1) + \\
& \mathbb{P}(A_2)\,\mathbb{P}(B|A_2) + \\
& \vdots \\
& \mathbb{P}(A_n)\,\mathbb{P}(B|A_n)
\end{aligned}
$$

The total probability theorem:

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)$$

Generalisation of $\mathbb{P}(B)$

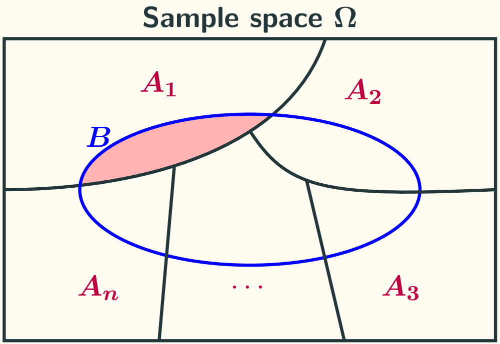


$$
\begin{aligned}
\mathbb{P}(B) =& \mathbb{P}[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)] \\
=& \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \cdots + \mathbb{P}(A_n \cap B) \\
=& \mathbb{P}(A_1)\,\mathbb{P}(B|A_1) + \\
& \mathbb{P}(A_2)\,\mathbb{P}(B|A_2) + \\
& \vdots \\
& \mathbb{P}(A_n)\,\mathbb{P}(B|A_n)
\end{aligned}
$$

The total probability theorem:

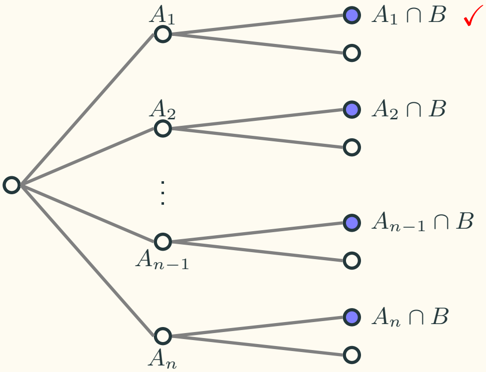
$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)$$

Generalisation of $\mathbb{P}(A|B)$



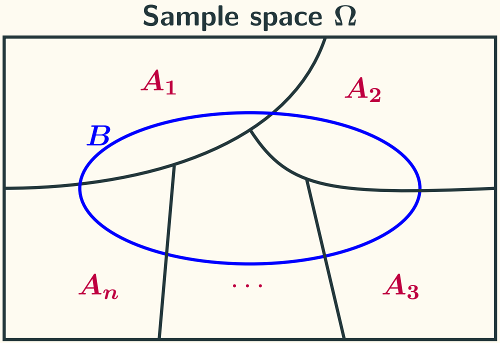
$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}$$

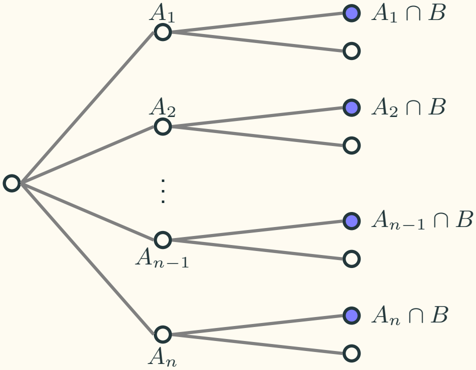
Generalisation of $\mathbb{P}\left(A|B\right)$



$$\begin{aligned}
\mathbb{P}\left(A_i|B\right) &= \frac{\mathbb{P}\left(A_i \cap B\right)}{\mathbb{P}\left(B\right)} \\
&= \frac{\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}{\mathbb{P}\left(B\right)} \\
&= \frac{\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}{\sum_{i=1}^{n}\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}
\end{aligned}$$

The Bayes Theorem





$$A_i \underset{\text{inference } \mathbb{P}(A_i|B)}{\overset{\text{causal effect } \mathbb{P}(B|A_i)}{\rightleftarrows}} B$$

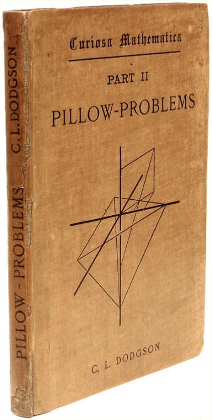
The Bayes' Theorem

$$\mathbb{P}\left(A_i|B\right) = \frac{\mathbb{P}\left(A_i \cap B\right)}{\mathbb{P}\left(B\right)} = \frac{\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}{\sum_{i=1}^{n}\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}$$

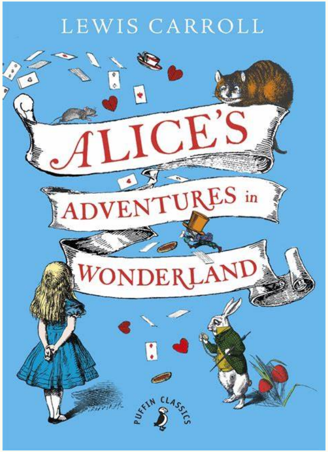
$\mathbb{P}\left(A_i\right)$: **prior probability**

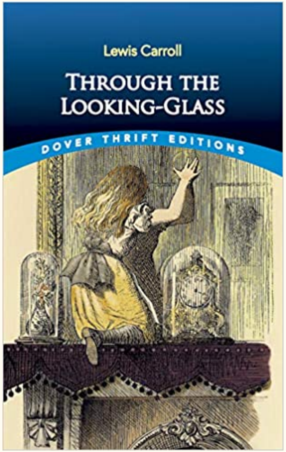
$\mathbb{P}\left(A_i|B\right)$: **posterior probability**

Lewis Carroll's Pillow Problems



by Charles Dodgson





Lewis Carroll's Pillow Problems

Question #5 (8th Sep 1887):

5. (19, 31)

A bag contains one counter, known to be either white or black. A white counter is put in, the bag shaken, and a counter drawn out, which proves to be white. What is now the chance of drawing a white counter? [8/9/87

Pedigree Analysis